A Dynamic Programming Approach for a Nonzero Sum Stochastic Differential Game Problem

Modelling and Simulation of Social-Behavioural Phenomena in Creative Societies (MSBC-2022)
2nd International European Conference
Vilnius, Lithuania

Gerhard Wilhelm Weber & Emel Savku

September 21-23, 2022

METU

Outline

Outline

Outline

Outline

Outline

Motivation

By regime-switches:

- The changes in the behaviors of financial markets, e.g., a shift from a volatile period to a calm period.
- The movements of financial markets between different states of economy, e.g., growth, recession.
- Discrete shifts from one regime to another by a change in economic policy, e.g., a shift in a monetary or an exchange rate policy.
- Different psychological preferences of investors.

Motivation

By regime-switches:

- The changes in the behaviors of financial markets, e.g., a shift from a volatile period to a calm period.
- The movements of financial markets between different states of economy, e.g., growth, recession.
- Discrete shifts from one regime to another by a change in economic policy, e.g., a shift in a monetary or an exchange rate policy.
- Different psychological preferences of investors.

Motivation

By regime-switches:

- The changes in the behaviors of financial markets, e.g., a shift from a volatile period to a calm period.
- The movements of financial markets between different states of economy, e.g., growth, recession.
- Discrete shifts from one regime to another by a change in economic policy, e.g., a shift in a monetary or an exchange rate policy.
- Different psychological preferences of investors.

Motivation

By regime-switches:

- The changes in the behaviors of financial markets, e.g., a shift from a volatile period to a calm period.
- The movements of financial markets between different states of economy, e.g., growth, recession.
- Discrete shifts from one regime to another by a change in economic policy, e.g., a shift in a monetary or an exchange rate policy.
- Different psychological preferences of investors.

Motivation

- Regulation changes with their deep Market Microstructure impact, e.g., tick size plot programs MiFID I & II *(Laruelle, Rosenbaum and Savku (2020))*, and similar ones by Tokyo SE and USA SEC.
- Macroeconomic indicators such as gross domestic product, purchase management index and sovereign credit rating.
- An activation by a major event, e.g., the bankruptcy of Lehman Brothers in September 2008 or the 1973 oil crisis.

Motivation

- Regulation changes with their deep Market Microstructure impact, e.g., tick size plot programs MiFID I & II *(Laruelle, Rosenbaum and Savku (2020))*, and similar ones by Tokyo SE and USA SEC.
- Macroeconomic indicators such as gross domestic product, purchase management index and sovereign credit rating.
- An activation by a major event, e.g., the bankruptcy of Lehman Brothers in September 2008 or the 1973 oil crisis.

Motivation

- Regulation changes with their deep Market Microstructure impact, e.g., tick size plot programs MiFID I & II *(Laruelle, Rosenbaum and Savku (2020))*, and similar ones by Tokyo SE and USA SEC.
- Macroeconomic indicators such as gross domestic product, purchase management index and sovereign credit rating.
- An activation by a major event, e.g., the bankruptcy of Lehman Brothers in September 2008 or the 1973 oil crisis.

Set up for Markov Switches

- Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} = (\mathscr{F}(t) : t \in [0,T])$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $(\alpha(t) : t \in [0,T])$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i, j = 1, 2, ..., D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i, j = 1, 2, ..., D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Set up for Markov Switches

- Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} = (\mathscr{F}(t) : t \in [0,T])$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $(\alpha(t) : t \in [0,T])$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i, j = 1, 2, ..., D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i, j = 1, 2, ..., D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Set up for Markov Switches

- Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} = (\mathscr{F}(t) : t \in [0,T])$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $(\alpha(t) : t \in [0,T])$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i,j = 1,2,...,D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i,j = 1,2,...,D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Set up for Markov Switches

- Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} = (\mathscr{F}(t) : t \in [0,T])$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $(\alpha(t) : t \in [0,T])$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i,j = 1,2,...,D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i,j = 1,2,...,D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Set up for Markov Switches

- Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} = (\mathscr{F}(t) : t \in [0,T])$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $(\alpha(t) : t \in [0,T])$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i, j = 1, 2, ..., D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i, j = 1, 2, ..., D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Set up for Markov Switches

- Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} = (\mathscr{F}(t) : t \in [0,T])$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $(\alpha(t) : t \in [0,T])$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i,j = 1, 2, ..., D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i,j = 1,2,...,D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} > 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Set up for Markov Switches

Elliot, Aggoun and Moore (1994) obtained the following semimartingale dynamics for the chain α:

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T \alpha(u)du + M(t),$$

where $M(t)$ is a an $\mathbb{R}^D$-valued $(\mathbb{F},\mathbb{P})$-martingale.
Let us define:

$$\begin{aligned} J^{ij}(t) &:= \sum_{0<s\leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s), e_j \rangle \\ &= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle ds + \int_0^t \langle \alpha(s-), e_i \rangle \langle dM(s), e_j \rangle . \end{aligned}$$

Let us define m_{ij} as follows:

$$m_{ij}(t) := \int_0^t \langle \alpha(s-), e_i \rangle \langle dM(s), e_j \rangle .$$

Set up for Markov Switches

For each fixed $j = 1, 2, ..., D$, let Φ_j be the number of jumps into state e_j up to time t. Then,

$$\Phi_j(t) := \sum_{i=1, i \neq j}^{D} J^{ij}(t)$$
$$= \sum_{i=1, i \neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds + \tilde{\Phi}_j(t),$$

where $\tilde{\Phi}_j(t) := \sum_{i=1, i \neq j}^{D} m_{ij}(t)$ and $\lambda_j(t) := \sum_{i=1, i \neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds$.

Hence, $\tilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t)$ is an $(\mathbb{F}, \mathbb{P})$-martingale.

Let us represent our model:

$$\begin{aligned} Y(t) &= b(t, Y(t), \alpha(t), u_1(t), u_2(t))dt \\ &\quad + \sigma(t, Y(t), \alpha(t), u_1(t), u_2(t))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t, Y(t-), \alpha(t-), u_1(t-), u_2(t-), z)\tilde{N}(dt, dz) \\ &\quad + \gamma(t, Y(t-), \alpha(t-), u_1(t-), u_2(t-))d\tilde{\Phi}(t), \; t \in [0, T], \qquad (1) \\ Y(0) &= y_0 \in \mathbb{R}^N, \end{aligned}$$

where $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is the compensated Poisson random measure, ν is the Lévy measure of the jump measure $N(\cdot, \cdot)$ and $u_1 \in \mathscr{U}_1$ and $u_2 \in \mathscr{U}_2$, which are non-empty subsets of $\mathbb{R}^N$.
The infinitesimal generator $\mathscr{L}^{u_1, u_2}$ for the system (1) is as in *Zhang, Elliott and Siu (2012)*:

DPP in Game Theory

$$
\begin{aligned}
\mathscr{L}^{u_1,u_2}[\phi(t,y,e_i)] =& \frac{\partial \phi}{\partial t}(t,y,e_i) + \sum_{k=1}^{N} \frac{\partial \phi}{\partial y_k}(t,y,e_i) b_k(t,y,e_i,u_1,u_2) \\
&+ \frac{1}{2} \sum_{k=1}^{N} \sum_{n=1}^{N} \frac{\partial^2 \phi}{\partial y_k \partial y_n}(t,y,e_i) \sum_{l=1}^{M} \sigma_{kl} \sigma_{nl}(t,y,e_i,u_1,u_2) \\
&+ \sum_{m=1}^{L} \int_{\mathbb{R}_0} \Big[\phi(t, y + \eta^{(m)}(t,y,e_i,u_1,u_2,z), e_i) - \phi(t,y,e_i) \\
&- \sum_{n=1}^{N} \frac{\partial \phi}{\partial y_n}(t,y,e_i) \eta_{nm}(t,y,e_i,u_1,u_2,z) \Big] \nu_m(dz) \\
&+ \sum_{j=1}^{D} \lambda_{ij} \Big[\phi(t, y + \gamma^{(j)}(t,y,e_i,u_1,u_2), e_j) - \phi(t,y,e_i) \\
&- \sum_{n=1}^{N} \frac{\partial \phi}{\partial y_n}(t,y,e_i) \gamma_{nj}(t,y,e_i,u_1,u_2) \Big].
\end{aligned}
$$

Lemma (*Dynkin Formula:*) Let $Y(\cdot) \in \mathbb{R}^N$ be a Markov regime-switching jump-diffusion process, G be an open subset of $\mathbb{R}^N$ and for all $e_i \in S$, $\phi(\cdot,\cdot,e_i) \in C^{1,2}(G) \cap C(\bar{G})$.
Let $\tau < \infty$ be a stopping time and $\tau \leq \tau_G := \inf\{t > 0, Y(t) \notin G\}$ and $Y(\tau) \in G$ a.s. for all $e_i \in S$.
Under some technical conditions *(Savku and Weber (2020), Lemma 3.1)*, we have

$$E^{t,y,e_i}[\phi(\tau, Y(\tau), \alpha(\tau))] = \phi(t,y,e_i) + E^{t,y,e_i}\left[\int_t^\tau \mathscr{L}[\phi(s, Y(s-), \alpha(s-))]\,ds\right]$$

for each $e_i \in S$.

Zero-Sum Stochastic Differential Game

Let

$$\tau_G = \inf\{t > 0, Y(t) \notin G\}$$

be the bankruptcy time, where $G \subset \mathbb{R}^N$ is an open set and represents the solvency region.

The performance (objective) functional:

$$J^{u_1,u_2}(t,y,e_i) = E\left[\int_t^{\tau_G} f(s,Y(s),\alpha(s),u_1(s),u_2(s))ds + g(Y(\tau_G),\alpha(\tau_G)\right].$$

Theorem (*Zero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1, 2, ..., D$, there exists a function $\phi \in C^{1,2}(G) \cap C(\bar{G})$ and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that

- $\mathscr{L}^{u_1,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1,u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*,u_2}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2) \geq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2^*) = 0$ for all $y \in G$ and $e_i \in S$.
- $Y^{u_1,u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi(t, Y^{u_1,u_2}(t), \alpha(t)) = g(Y^{u_1,u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ y \in G$.
- The family $\{\phi(\tau, Y^{u_1,u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1,u_2) \in \Theta_1 \times \Theta_2$.

Theorem (*Zero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1,2,...,D$, there exists a function $\phi \in C^{1,2}(G) \cap C(\bar{G})$ and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that

- $\mathscr{L}^{u_1,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1,u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*,u_2}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2) \geq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2^*) = 0$ for all $y \in G$ and $e_i \in S$.
- $Y^{u_1,u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi(t, Y^{u_1,u_2}(t), \alpha(t)) = g(Y^{u_1,u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ y \in G$.
- The family $\{\phi(\tau, Y^{u_1,u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1,u_2) \in \Theta_1 \times \Theta_2$.

Theorem (*Zero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1, 2, ..., D$, there exists a function $\phi \in C^{1,2}(G) \cap C(\bar{G})$ and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that

- $\mathscr{L}^{u_1, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1, u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2) \geq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2^*) = 0$ for all $y \in G$ and $e_i \in S$.
- $Y^{u_1, u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi(t, Y^{u_1, u_2}(t), \alpha(t)) = g(Y^{u_1, u_2}(\tau_G), \alpha(\tau_G))1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ y \in G$.
- The family $\{\phi(\tau, Y^{u_1, u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1, u_2) \in \Theta_1 \times \Theta_2$.

Theorem (*Zero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1, 2, ..., D$, there exists a function $\phi \in C^{1,2}(G) \cap C(\bar{G})$ and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that

- $\mathscr{L}^{u_1, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1, u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2) \geq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2^*) = 0$ for all $y \in G$ and $e_i \in S$.
- $Y^{u_1, u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi(t, Y^{u_1, u_2}(t), \alpha(t)) = g(Y^{u_1, u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ y \in G$.
- The family $\{\phi(\tau, Y^{u_1, u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1, u_2) \in \Theta_1 \times \Theta_2$.

Theorem (*Zero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1, 2, ..., D$, there exists a function $\phi \in C^{1,2}(G) \cap C(\bar{G})$ and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that

- $\mathscr{L}^{u_1, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1, u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1, \ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2) \geq 0$ for all $u_2 \in \mathscr{U}_2, \ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2^*) = 0$ for all $y \in G$ and $e_i \in S$.
- $Y^{u_1, u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi(t, Y^{u_1, u_2}(t), \alpha(t)) = g(Y^{u_1, u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1, u_2) \in \Theta_1 \times \Theta_2, \ y \in G$.
- The family $\{\phi(\tau, Y^{u_1, u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1, u_2) \in \Theta_1 \times \Theta_2$.

Zero-Sum Stochastic Differential Game

Then,

$$\phi(t,y,e_i) = V(t,y,e_i) = \sup_{u_1\in\Theta_1}\left(\inf_{u_2\in\Theta_2} J^{u_1,u_2}(t,y,e_i)\right)$$
$$= \inf_{u_2\in\Theta_2}\left(\sup_{u_1\in\Theta_1} J^{u_1,u_2}(t,y,e_i)\right)$$
$$= \sup_{u_1\in\Theta_1} J^{u_1,u_2^*}(t,y,e_i) = \inf_{u_2\in\Theta_2} J^{u_1^*,u_2}(t,y,e_i),$$
$$= J^{u_1^*,u_2^*}(t,y,e_i), \qquad y\in G,\; e_i\in S,$$

and (u_1^*, u_2^*) is a saddle point (a Markovian optimal control) of the Zero-Sum Stochastic Differential Game.

Market vs Trader

- There is no transaction cost.
- Infinite divisible assets are allowed.
- Information is symmetric.
- The mean rate of the risky asset $\theta(\cdot)$ is not given a priori.
- $\theta(\cdot)$ is considered as a consequence of the portfolio choice $\pi(\cdot)$ of the investor.
- Min-Max problem between the trader and the market.

Market vs Trader

- There is no transaction cost.
- Infinite divisible assets are allowed.
- Information is symmetric.
- The mean rate of the risky asset $\theta(\cdot)$ is not given a priori.
- $\theta(\cdot)$ is considered as a consequence of the portfolio choice $\pi(\cdot)$ of the investor.
- Min-Max problem between the trader and the market.

Market vs Trader

- There is no transaction cost.
- Infinite divisible assets are allowed.
- Information is symmetric.
- The mean rate of the risky asset $\theta(\cdot)$ is not given a priori.
- $\theta(\cdot)$ is considered as a consequence of the portfolio choice $\pi(\cdot)$ of the investor.
- Min-Max problem between the trader and the market.

Market vs Trader

- There is no transaction cost.
- Infinite divisible assets are allowed.
- Information is symmetric.
- The mean rate of the risky asset $\theta(\cdot)$ is not given a priori.
- $\theta(\cdot)$ is considered as a consequence of the portfolio choice $\pi(\cdot)$ of the investor.
- Min-Max problem between the trader and the market.

Market vs Trader

- There is no transaction cost.
- Infinite divisible assets are allowed.
- Information is symmetric.
- The mean rate of the risky asset $\theta(\cdot)$ is not given a priori.
- $\theta(\cdot)$ is considered as a consequence of the portfolio choice $\pi(\cdot)$ of the investor.
- Min-Max problem between the trader and the market.

Market vs Trader

- There is no transaction cost.
- Infinite divisible assets are allowed.
- Information is symmetric.
- The mean rate of the risky asset $\theta(\cdot)$ is not given a priori.
- $\theta(\cdot)$ is considered as a consequence of the portfolio choice $\pi(\cdot)$ of the investor.
- Min-Max problem between the trader and the market.

The risk-free bond for instantaneous borrowing or lending at risk-free rate:

$$
\begin{aligned}
dS_0(t) = &\ S_0(t) r(t, \alpha(t-)) dt, \qquad t \in [0,T], \\
S_0(0) = &\ s_0 > 0.
\end{aligned}
$$

$r(t, e_i) = r(t)$ for any $e_i \in S,\ i = 1, 2, ..., D$ and $t \in [0,T]$.
The dynamics of risky asset:

$$
\begin{aligned}
dS(t) = & S(t-)\Big(\theta(t)dt + \sigma(t, \alpha(t-))dW(t) + \int_{\mathbb{R}_0} \eta(t, \alpha(t-), z)\tilde{N}(dt, dz) \\
& + \gamma(t, \alpha(t-))d\tilde{\Phi}(t)\Big), \qquad t \in [0,T], \\
S(0) = & s > 0.
\end{aligned}
$$

Market vs Trader

Hence, the wealth process of the investor is described by:

$$dX^{\pi,\theta}(t) = X(t-)\Big[\{(1-\pi(t))r(t) + \pi(t)\theta(t)\}\,dt + \pi(t)\sigma(t,\alpha(t))dW(t)$$
$$+\pi(t-)\int_{\mathbb{R}_0}\eta(t,\alpha(t-),z)\tilde{N}(dt,dz) + \pi(t-)\gamma(t,\alpha(t-))d\tilde{\Phi}(t)\Big],\ t\in[0,T],$$
$$X^{\pi,\theta}(0) = x > 0,$$

where $(\theta,\pi)\in\Theta_1\times\Theta_2$.

Our problem is to find the saddle point of this game $(\pi^*, \theta^*) \in \Theta_1 \times \Theta_2$ and the value function $V(t, x, e_i)$ for all $t \in [0, T]$ and $e_i \in S$ such that

$$V(t,x,e_i) = \inf_{\theta \in \Theta_1} \left(\sup_{\pi \in \Theta_2} E^{t,x,e_i} \left[U(X^{\pi,\theta}(T)) \right] \right) = E^{t,x,e_i} \left[U(X^{\pi^*,\theta^*}(T)) \right].$$

We provide the HJBI equation for the value function V in the form of:

$$V(t,x,e_i) = U(x \exp(\int_t^T r(s)ds)).$$

Then, by applying infinitesimal generator on V, we obtain:

Market vs Trader

$$
\begin{aligned}
&\mathscr{L}^{\pi,\theta}[\phi(t,y,e_i)] = -U'(x\exp(\int_t^T r(s)ds))x\exp(\int_t^T r(s)ds)r(t) \\
&+x((1-\pi)r(t)+\pi\theta)U'(x\exp(\int_t^T r(s)ds))\exp(\int_t^T r(s)ds) \\
&+\frac{1}{2}U''(x\exp(\int_t^T r(s)ds))(\exp(\int_t^T r(s)ds))^2\pi^2\sigma^2(t,e_i)x^2 \\
&+\int_{\mathbb{R}_0}\Big\{U((x+x\pi\eta(t,e_i,z))\exp(\int_t^T r(s)ds)) - U(x\exp(\int_t^T r(s)ds)) \\
&-U'(x\exp(\int_t^T r(s)ds))\exp(\int_t^T r(s)ds)(\pi x\eta(t,e_i,z))\Big\}\nu(dz) \\
&+\sum_{j=1}^{D}\lambda_{ij}\Big\{U((x+x\pi\gamma^j(t,e_i))\exp(\int_t^T r(s)ds)) - U(x\exp(\int_t^T r(s)ds)) \\
&-U'(x\exp(\int_t^T r(s)ds))\exp(\int_t^T r(s)ds)(\pi x\gamma^j(t,e_i))\Big\}.
\end{aligned}
$$

Results

By applying FOC, we obtain that the optimal fraction of the trader's wealth held in the risky asset $\pi^*(\theta) = 0$ and consequently, $\theta^*(t) = r(t), \ \ t \in [0, T]$. Hence,

- there is no trade,
- this solution establishes a *dynamic equilibrium* between the market and the investor,
- this result is compatible with the *fundamental equilibrium of risk-neutral asset pricing*.

Results

By applying FOC, we obtain that the optimal fraction of the trader's wealth held in the risky asset $\pi^*(\theta) = 0$ and consequently, $\theta^*(t) = r(t), \;\; t \in [0,T]$. Hence,

- there is no trade,
- this solution establishes a *dynamic equilibrium* between the market and the investor,
- this result is compatible with the *fundamental equilibrium of risk-neutral asset pricing*.

Results

By applying FOC, we obtain that the optimal fraction of the trader's wealth held in the risky asset $\pi^*(\theta) = 0$ and consequently, $\theta^*(t) = r(t), \;\; t \in [0, T]$. Hence,

- there is no trade,
- this solution establishes a *dynamic equilibrium* between the market and the investor,
- this result is compatible with the *fundamental equilibrium of risk-neutral asset pricing*.

Nonzero-Sum Stochastic Differential Game

Let $u_1 \in \Theta_1$ and $u_2 \in \Theta_2$ be two admissible control processes for Player 1 and Player 2, respectively.
The performance functionals (payoff) to Player number *k*, k=1,2:

$$J_k^{u_1,u_2}(t,y,e_i) = E^{t,y,e_i}\left[\int_t^{\tau_G} f_k(s,Y(s),\alpha(s),u_1(s),u_2(s))ds + g_k(Y(\tau_G),\alpha(\tau_G))\right] \quad (2)$$

for each $e_i \in S$.

Nonzero-Sum Stochastic Differential Game

Let us assume that for the optimal strategy of Player 2, $u_2^* \in \Theta_2$, the best response of Player 1 satisfies

$$J_1^{u_1,u_2^*}(t,y,e_i) \leq J_1^{u_1^*,u_2^*}(t,y,e_i) \text{ for all } u_1 \in \Theta_1,\ e_i \in S,\ y \in G,$$

and for the optimal strategy of Player 1, $u_1^* \in \Theta_1$, the best response of Player 2 satisfies

$$J_2^{u_1^*,u_2}(t,y,e_i) \leq J_2^{u_1^*,u_2^*}(t,y,e_i) \text{ for all } u_2 \in \Theta_2,\ e_i \in S,\ y \in G.$$

Then, the pair of optimal control processes $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ is called a Nash equilibrium for the stochastic differential game of Equations (1) and (2).

Nonzero-Sum Stochastic Differential Game

Theorem (*Nonzero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1,2,...,D$, there exists functions $\phi_k \in C^{1,2}(G) \cap C(\bar{G}),\ k = 1,2$, and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that the following conditions are fulfilled:

- $\mathscr{L}^{u_1,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1,u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*,u_2}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2) \leq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $Y^{u_1,u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi_k(t, Y^{u_1,u_2}(t), \alpha(t)) = g_k(Y^{u_1,u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ y \in G,\ k = 1,2.$
- The family $\{\phi_k(\tau, Y^{u_1,u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ k = 1,2.$

Nonzero-Sum Stochastic Differential Game

Theorem (*Nonzero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1,2,...,D$, there exists functions $\phi_k \in C^{1,2}(G) \cap C(\bar{G}),\ k = 1,2$, and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that the following conditions are fulfilled:

- $\mathscr{L}^{u_1,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1,u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*,u_2}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2) \leq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $Y^{u_1,u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi_k(t, Y^{u_1,u_2}(t), \alpha(t)) = g_k(Y^{u_1,u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ y \in G,\ k = 1,2.$
- The family $\{\phi_k(\tau, Y^{u_1,u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ k = 1,2.$

Theorem (*Nonzero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1, 2, ..., D$, there exists functions $\phi_k \in C^{1,2}(G) \cap C(\bar{G}),\ k = 1, 2$, and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that the following conditions are fulfilled:

- $\mathscr{L}^{u_1, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1, u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2) \leq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $Y^{u_1, u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi_k(t, Y^{u_1, u_2}(t), \alpha(t)) = g_k(Y^{u_1, u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ y \in G,\ k = 1, 2$.
- The family $\{\phi_k(\tau, Y^{u_1, u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ k = 1, 2$.

Theorem (*Nonzero-Sum Stochastic Differential Game*):
Suppose that for each $e_i, i = 1, 2, ..., D$, there exists functions $\phi_k \in C^{1,2}(G) \cap C(\bar{G}),\ k = 1, 2$, and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that the following conditions are fulfilled:

- $\mathscr{L}^{u_1, u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1, u_2^*) \leq 0$ for all $u_1 \in \mathscr{U}_1,\ y \in G$ and $e_i \in S$.
- $\mathscr{L}^{u_1^*, u_2}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2) \leq 0$ for all $u_2 \in \mathscr{U}_2,\ y \in G$ and $e_i \in S$.
- $Y^{u_1, u_2}(\tau_G) \in \partial G$ a.s. on $\{\tau_G < \infty\}$ and $\lim_{t \to \tau_G^-} \phi_k(t, Y^{u_1, u_2}(t), \alpha(t)) = g_k(Y^{u_1, u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ a.s. for all $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ y \in G,\ k = 1, 2$.
- The family $\{\phi_k(\tau, Y^{u_1, u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathscr{T}\}}$ is uniformly integrable for all $y \in G$ and $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ k = 1, 2$.

Then, for all $y \in G,\ e_i \in S,\ (u_1^*, u_2^*)$ is a Nash equilibrium for the game (2) subject to the goals of the system (1) such that

$$\phi_1(t,y,e_i) = V_1(t,y,e_i) = \sup_{u_1 \in \Theta_1} J_1^{u_1,u_2^*}(t,y,e_i) = J_1^{u_1^*,u_2^*}(t,y,e_i),$$

$$\phi_2(t,y,e_i) = V_2(t,y,e_i) = \sup_{u_2 \in \Theta_2} J_2^{u_1^*,u_2}(t,y,e_i) = J_2^{u_1^*,u_2^*}(t,y,e_i).$$

Optimal Strategies For Two Companies

- There are two companies in collaboration.
- A certain percentage of the shares of Company 1 belongs to the Company 2.
- Company 1 is designed to be regarded as a semiconductor industry.
- A certain percentage of the terminal payoff of Company 1 affects the terminal payoff of Company 2, and vice versa.

Optimal Strategies For Two Companies

- There are two companies in collaboration.
- A certain percentage of the shares of Company 1 belongs to the Company 2.
- Company 1 is designed to be regarded as a semiconductor industry.
- A certain percentage of the terminal payoff of Company 1 affects the terminal payoff of Company 2, and vice versa.

Optimal Strategies For Two Companies

- There are two companies in collaboration.
- A certain percentage of the shares of Company 1 belongs to the Company 2.
- Company 1 is designed to be regarded as a semiconductor industry.
- A certain percentage of the terminal payoff of Company 1 affects the terminal payoff of Company 2, and vice versa.

Optimal Strategies For Two Companies

- There are two companies in collaboration.
- A certain percentage of the shares of Company 1 belongs to the Company 2.
- Company 1 is designed to be regarded as a semiconductor industry.
- A certain percentage of the terminal payoff of Company 1 affects the terminal payoff of Company 2, and vice versa.

Optimal Strategies For Two Companies

The risky assets with prices $S_1(\cdot)$ for Investor 1 and $S_2(\cdot)$ for Investor 2:

$$\begin{aligned} dS_m(t) =& S_m(t-)\Big(\mu_m(t,\alpha(t-))dt + \sigma_m(t,\alpha(t-))dW(t) \\ &+ \int_{\mathbb{R}_0} \eta_m(t,\alpha(t-),z)\tilde{N}(dt,dz)\Big), \qquad t \in [0,T], \\ S_m(0) =& s_m > 0, \qquad m = 1,2. \end{aligned}$$

The risk-free bond $S_0(\cdot)$:

$$\begin{aligned} dS_0(t) &= S_0(t)r(t,\alpha(t-))dt, \qquad t \in [0,T], \\ S_0(0) &= s_0 > 0. \end{aligned}$$

Optimal Strategies For Two Companies

The dynamics of the wealth processes of each investor are as follows:

$$
\begin{aligned}
dX_m(t) &= X_m(t-)\Big(\pi_m(t)\mu_m(t,\alpha(t-)) + (1-\pi_m(t))r(t,\alpha(t-))\Big)dt \\
&\quad + X_m(t-)\pi_m(t)\Big(\sigma_m(t,\alpha(t-))dW(t) + \int_{\mathbb{R}_0} \eta_m(t,\alpha(t-),z)\tilde{N}(dt,dz)\Big), \\
&\qquad t \in [0,T], \\
X_m(0) &= x_m > 0,\ m = 1,2, \qquad (3)
\end{aligned}
$$

where $\pi_1 \in \mathscr{U}_1$ and $\pi_2 \in \mathscr{U}_2$ are admissible.

Optimal Strategies For Two Companies

One investor's final saving is as a factor of sensitivity for the other investor and vice versa:

$$J_1(t,x_1,x_2,e_i,\pi_1,\pi_2) = E^{t,x_1,x_2,e_i}\Big[\gamma_1 X_1(T) X_2(T)\Big],$$

$$J_2(t,x_1,x_2,e_i,\pi_1,\pi_2) = E^{t,x_1,x_2,e_i}\Big[\gamma_2 X_1(T) X_2(T)\Big],$$

where $\gamma_1, \gamma_2 \in \mathbb{R}^+$. Then, our problem is to find $(\pi_1^*, \pi_2^*) \in \Theta_1 \times \Theta_2$ and

$$V_1(t,x_1,x_2,e_i) = \sup_{\pi_1 \in \Theta_1} J_1(t,x_1,x_2,e_i,\pi_1,\pi_2^*) = J_1^{\pi_1^*,\pi_2^*}(t,x_1,x_2,e_i),$$

$$V_2(t,x_1,x_2,e_i) = \sup_{\pi_2 \in \Theta_2} J_2(t,x_1,x_2,e_i,\pi_1^*,\pi_2) = J_2^{\pi_1^*,\pi_2^*}(t,x_1,x_2,e_i).$$

Optimal Strategies For Two Companies

We can re-state our problem for each investor by the following HJBI equations:

$$\sup_{\pi_1 \in \Theta_1} \left\{ \mathscr{L}^{\pi_1, \pi_2^*} [\phi_1(t, x_1, x_2, e_i)] \right\} = 0,$$
$$\phi_1(T, x_1, x_2, e_i) = \gamma_1 x_1 x_2, \tag{4}$$

and

$$\sup_{\pi_2 \in \Theta_2} \left\{ \mathscr{L}^{\pi_1^*, \pi_2} [\phi_2(t, x_1, x_2, e_i)] \right\} = 0,$$
$$\phi_2(T, x_1, x_2, e_i) = \gamma_2 x_1 x_2, \tag{5}$$

for all $e_i \in S$. A value function of the form:

$$V_m(t, x_1, x_1, e_i) = k_m(t, e_i) x_1 x_2$$

.

Optimal Strategies For Two Companies

Hence, the optimal portfolio strategies for Investor 1 and 2 are as follows:

$$\pi_1^* = \frac{r(t,e_i) - \mu_2(t,e_i)}{\sigma_1(t,e_i)\sigma_2(t,e_i) + \int_{\mathbb{R}_0} \eta_1(t,e_i,z)\eta_2(t,e_i,z)\nu(dz)} \tag{6}$$

and

$$\pi_2^* = \frac{r(t,e_i) - \mu_1(t,e_i)}{\sigma_1(t,e_i)\sigma_2(t,e_i) + \int_{\mathbb{R}_0} \eta_1(t,e_i,z)\eta_2(t,e_i,z)\nu(dz)}, \tag{7}$$

for all $e_i \in S$.

If we consider π_1^* and π_2^* in Equations (4) and (5), then for all $e_i \in S$, we obtain D-coupled, linear ordinary differential equations (linear ODEs):

$$k_m^{'}(t,e_i) + h(t,e_i)k_m(t,e_i) + \sum_{j=1}^{D} \lambda_{ij}(k_m(t,e_j) - k_m(t,e_i)) = 0,$$

$$k_m(T,e_i) = \gamma_m > 0, \qquad \text{for} \quad m = 1,2, \tag{8}$$

where

$$h(t,e_i) := \pi_1^* \mu_1(t,e_i) + (1-\pi_1^*)r(t,e_i) + \pi_2^* \mu_2(t,e_i) + (1-\pi_2^*)r(t,e_i)$$
$$+ \pi_1^* \pi_2^* \left(\sigma_1(t,e_i)\sigma_2(t,e_i) + \int_{\mathbb{R}_0} \eta_1(t,e_i,z)\eta_2(t,e_i,z)\nu(dz) \right).$$

Optimal Strategies For Two Companies

By applying the classical procedure of Feynman-Kac representation, the value functions of each investor:

$$V_m(t, x_1, x_1, e_i) = k_m(t, e_i) x_1 x_2,$$

where

$$k_m(t, \alpha(t)) = \gamma_m E\left[\exp\left(\int_t^T h(s, \alpha(s)) ds\right) | \alpha(t) = e_i\right], \qquad \text{for} \quad m = 1, 2.$$

Optimal Strategies For Two Companies

- The relation between the prices of petroleum and the dollar under the political strategy switches in a petroleum region.
- The interaction between the electricity prices and the prices of an agricultural product from a greenhouse (climate switches or some natural, catastrophic events like flood, frosty weather, etc.).
- Each strategy or catastrophic event may lead investors to different states of psychological tendency.

Optimal Strategies For Two Companies

- The relation between the prices of petroleum and the dollar under the political strategy switches in a petroleum region.
- The interaction between the electricity prices and the prices of an agricultural product from a greenhouse (climate switches or some natural, catastrophic events like flood, frosty weather, etc.).
- Each strategy or catastrophic event may lead investors to different states of psychological tendency.

Optimal Strategies For Two Companies

- The relation between the prices of petroleum and the dollar under the political strategy switches in a petroleum region.
- The interaction between the electricity prices and the prices of an agricultural product from a greenhouse (climate switches or some natural, catastrophic events like flood, frosty weather, etc.).
- Each strategy or catastrophic event may lead investors to different states of psychological tendency.

Special Case

$S = \{e_1, e_2\}$: a "*good*" and "*bad*" economy, or a "*bear*" and "*bull*" market.

The rate matrix of the Markov chain α can be represented as follows:

$$\begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.$$

Special Case

Thus, by Equation (8), we obtain 2-coupled linear ODEs with terminal values:

$$k_1'(t,e_1) + d^1 k_1(t,e_1) + \lambda(k_1(t,e_2) - k_1(t,e_1)) = 0, \qquad k_1(T,e_1) = \gamma_1,$$
$$k_1'(t,e_2) + d^2 k_1(t,e_2) + \lambda(k_1(t,e_1) - k_1(t,e_2)) = 0, \qquad k_1(T,e_2) = \gamma_1,$$

and

$$k_2'(t,e_1) + d^1 k_2(t,e_1) + \lambda(k_2(t,e_2) - k_2(t,e_1)) = 0, \qquad k_2(T,e_1) = \gamma_2,$$
$$k_2'(t,e_2) + d^2 k_2(t,e_2) + \lambda(k_2(t,e_1) - k_2(t,e_2)) = 0, \qquad k_2(T,e_2) = \gamma_2,$$

where $h(t,e_1) = d^1$ and $h(t,e_2) = d^2$.

Special Case

By classical methods of solving ODEs, we receive the following solutions explicitly:

$$k_1(T,e_1) = C_3 e^{-p_1(T-t)} + C_4 e^{-p_2(T-t)}, \qquad k_1(T,e_1) = \gamma_1,$$
$$k_1(T,e_2) = C_1 e^{-p_1(T-t)} + C_2 e^{-p_2(T-t)}, \qquad k_1(T,e_2) = \gamma_1,$$

where

$$C_1 = -\gamma_1 \frac{(d^2+p_2)}{p_1-p_2}, \qquad C_2 = \gamma_1 \frac{(d^2+p_1)}{p_1-p_2},$$
$$C_3 = -\gamma_1 \frac{(d^1+p_2)}{p_1-p_2}, \qquad C_4 = \gamma_1 \frac{(d^1+p_1)}{p_1-p_2}.$$

Special Case

Similarly,

$$k_2(T,e_2) = C_5 e^{-p_1(T-t)} + C_6 e^{-p_2(T-t)}, \qquad k_2(T,e_2) = \gamma_2,$$
$$k_2(T,e_1) = C_7 e^{-p_1(T-t)} + C_8 e^{-p_2(T-t)}, \qquad k_2(T,e_1) = \gamma_2,$$

where

$$C_5 = -\gamma_2 \frac{(d^2 + p_2)}{p_1 - p_2}, \qquad C_6 = \gamma_2 \frac{(d^2 + p_1)}{p_1 - p_2},$$
$$C_7 = -\gamma_2 \frac{(d^1 + p_2)}{p_1 - p_2}, \qquad C_8 = \gamma_2 \frac{(d^1 + p_1)}{p_1 - p_2}.$$

By Equations (6)-(7):

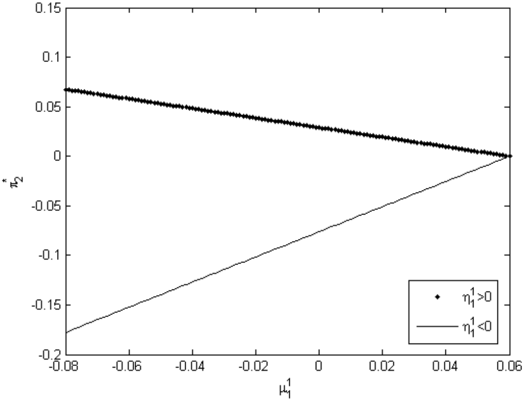


Figure: Optimal π_2 in BE against μ_1^1 for different η_1^1.

Comparative Results

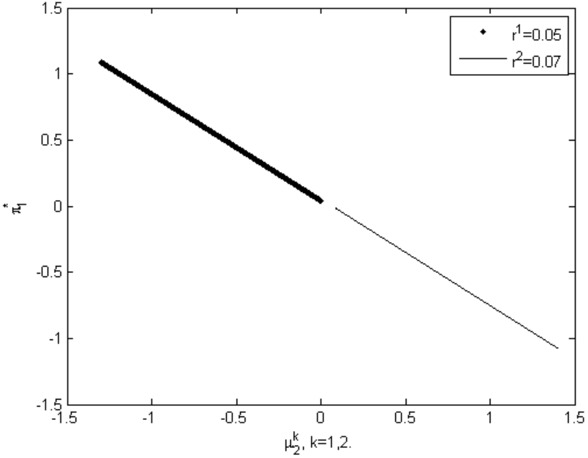
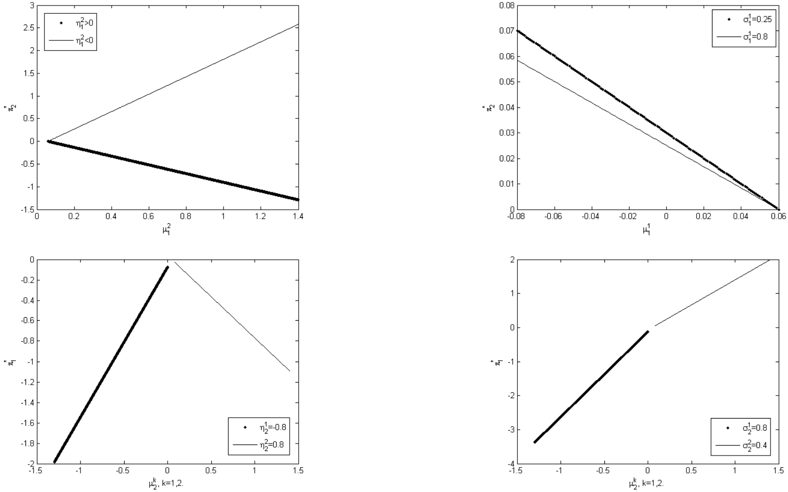


Figure: Optimal π_1 in GE-BE against $\mu_2^k,\ k=1,2$, for different $r^k,\ k=1,2$.



$\eta_1^2>0$
$\eta_1^2<0$
π_2^*
μ_1^2
$\sigma_1^1=0.25$
$\sigma_1^1=0.8$
π_2^*
μ_1^1
$\eta_2^1=-0.8$
$\eta_2^2=0.8$
π_1^*
μ_2^k, k=1,2.
$\sigma_2^1=0.8$
$\sigma_2^2=0.4$
π_1^*
μ_2^k, k=1,2.

THANK YOU VERY MUCH FOR YOUR ATTENTION...

References and Selected Works

- R. J. Elliott, L. Aggoun, and J. B. Moore, Hidden Markov Models: Estimation and Control, Springer, New York, 1994.
- S. Laruelle, M. Rosenbaum and E. Savku, Assessing MiFID 2 regulation on tick sizes: A transaction costs analysis viewpoint. Market Microstructure and Liquidity (2020).
- S. Mataramvura and B. Øksendal, Risk minimizing portfolios and HJBI equations or stochastic differential games, Stochastics An International Journal of Probability and Stochastic Processes, 80(4), 317-337, 2008.

References and Selected Works

- R. J. Elliott, L. Aggoun, and J. B. Moore, Hidden Markov Models: Estimation and Control, Springer, New York, 1994.
- S. Laruelle, M. Rosenbaum and E. Savku, Assessing MiFID 2 regulation on tick sizes: A transaction costs analysis viewpoint. Market Microstructure and Liquidity (2020).
- S. Mataramvura and B. Øksendal, Risk minimizing portfolios and HJBI equations or stochastic differential games, Stochastics An International Journal of Probability and Stochastic Processes, 80(4), 317-337, 2008.

References and Selected Works

- R. J. Elliott, L. Aggoun, and J. B. Moore, Hidden Markov Models: Estimation and Control, Springer, New York, 1994.
- S. Laruelle, M. Rosenbaum and E. Savku, Assessing MiFID 2 regulation on tick sizes: A transaction costs analysis viewpoint. Market Microstructure and Liquidity (2020).
- S. Mataramvura and B. Øksendal, Risk minimizing portfolios and HJBI equations or stochastic differential games, Stochastics An International Journal of Probability and Stochastic Processes, 80(4), 317-337, 2008.

References and Selected Works

- G. Peskir and J. Shorish, Market forces and dynamic asset pricing, Stochastic Analysis and Applications, 20(5), 1027-1082, 2002.
- E. Savku and G.-W. Weber, Stochastic differential games for optimal investment problems in a Markov regime-switching jump-diffusion market. Ann Oper Res (2020) DOI: 10.1007/s10479-020-03768-5.

References and Selected Works

- G. Peskir and J. Shorish, Market forces and dynamic asset pricing, Stochastic Analysis and Applications, 20(5), 1027-1082, 2002.
- E. Savku and G.-W. Weber, Stochastic differential games for optimal investment problems in a Markov regime-switching jump-diffusion market. Ann Oper Res (2020) DOI: 10.1007/s10479-020-03768-5.