Equilibrium in the strategic bargaining

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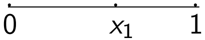
Outline

1. Cake cutting
2. Rubinstein approach
3. Choosing the meeting time
4. Stochastic design
5. Tournament solution

Resource distribution and bargaining

The **Rubinstein**'s bargaining model [1982], provided a convenient tool for solving game-theoretic problems about bidding by two persons with alternating offers on an infinite time axis. The main feature was the discounting factor δ, the closer δ is to 0, the more impatient the players are and the faster they will agree to any offer. On the contrary, if the value δ is close to 1, then the players are patient and will negotiate until they come to the most favorable offer for them. In [**Baron, Ferejohn**, 1989], a model of sequential with a majority rule was proposed. The game that was reviewed is a standard game "split the dollar", in which n players, whose turn is chosen randomly, make suggestions on how to divide a pie of a fixed size, and agreement requires the consent of a simple majority. It is shown that a sub-game perfect equilibrium exists in the class of stationary strategies. Then, articles [**Eraslan, 2002; Cho, Duggan, 2003; Banks, Duggan, 2006; Predtetchinski, 2011; Cardona, Ponsati, 2007, 2011**] were devoted to the expansion of this direction for different applied problems.

Sequential bargaining. Rubinstein approach



The **players** *I* **and** *II* make sequential offers for cake cutting and the process finishes, as one of them accepts the offer of another.
Discounting: At shot 1, the size of cake is 1; at shot 2, it makes $\delta < 1$, at shot 3, δ^2, and so on.
At shot 1 and all subsequent odd shots, player *I* makes his offer (x_1, x_2), where x_1 indicates the share of cake for player *I*, and x_2 means the share for player *II*.
We will seek for a **subgame-perfect equilibrium**, i.e., an equilibrium in all subgames of this game.

Apply the backward induction technique.
We begin with the case of three shots. The scheme of negotiations is as follows.

1. Player I makes the offer $(x_1, 1 - x_1)$, where $x_1 \leq 1$. If player II agrees, the game finishes—players I and II receive the payoffs x_1 and $1 - x_1$, respectively. Otherwise, the game continues to the next shot.
2. Player II makes the new offer $(x_2, 1 - x_2)$, where $x_2 \leq 1$. If player I accepts it, the game finishes. Players I and II gain the payoffs x_2 and $1 - x_2$, respectively. If player I rejects the offer, the game continues to shot 3.
3. The game finishes such that players I and II get the payoffs y and $1 - y$, respectively ($y \leq 1$ is a given value). In the sequel, we will establish the following fact. This value has no impact on the optimal solution under a sufficiently large duration of negotiations.

Sequential bargaining. Rubinstein approach

To find a subgame-perfect equilibrium, apply the backward induction method. Suppose that negotiations run at shot 2 and player *II* makes an offer. He should make a certain offer x_2 to player *I* such that his payoff is higher than at shot 3. Due to the discounting effect, the payoff of player *I* at the last shot constitutes δy. Therefore, player *I* agrees with the offer x_2, iff

$$x_2 \geq \delta y.$$

On the other hand, if player *II* offers $x_2 = \delta y$ to the opponent, his payoff becomes $1 - \delta y$. However, if his offer appears nonbeneficial to player *I*, the game continues to shot 3 and player *II* gains $\delta(1 - y)$ (recall the discounting effect). Note that $\delta(1 - y) < 1 - \delta y$. Hence, the optimal offer of player *II* is

$$x_2^* = \delta y.$$

Now, imagine that negotiations run at shot 1 and offer belongs to player I. He knows the opponent's offer at the next shot. And so, player I should make an offer $1 - x_1$ to the opponent such that the latter's payoff is at least the same as at shot 2: $\delta(1 - x_2^*) = \delta(1 - \delta y)$. Player II feels satisfied if $1 - x_1 \geq \delta(1 - \delta y)$ or

$$x_1 \leq 1 - \delta(1 - \delta y).$$

Thus, the following offer of player I is surely accepted by his opponent: $x_1 = 1 - \delta(1 - \delta y)$. If player I offers less, he receives the discounted payoff at shot 2: $\delta x_2^* = \delta^2 y$. Still, this quantity turns out smaller than $1 - \delta(1 - \delta y)$. Therefore, the optimal offer of player I forms

$$x_1^* = 1 - \delta(1 - \delta y) = 1 - \delta + \delta^2 y$$

and it will be accepted by player II. The sequence $\{x_1^*, x_2^*\}$ represents a subgame-perfect equilibrium in this negotiation game with three shots.

Arguing by induction, a subgame-perfect equilibrium in the negotiation game with n shots is such that

$$x_1^n = 1 - \delta + \delta^2 - ... + (-\delta)^{n-2} + (-\delta)^{n-1} y. \tag{1}$$

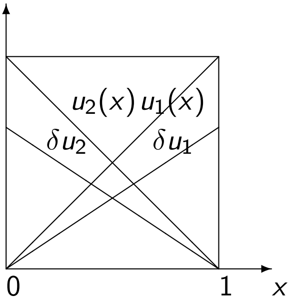
For large n, the last summand in (1), containing y, becomes infinitesimal, whereas the optimal offer of player I equals $x_1^* = 1/(1+\delta)$.

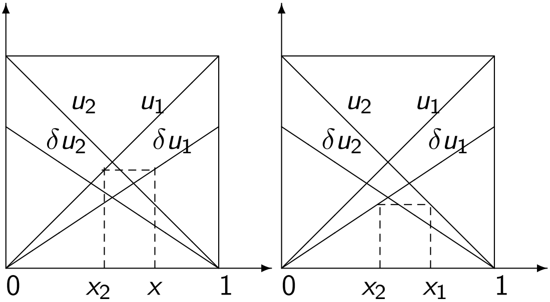
Theorem. *The sequential negotiation game of two players admits the subgame-perfect equilibrium*

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right).$$

Sequential bargaining. Utility functions

Two players I and II. The utilities are described by continuous unimodal functions $u_1(x) = x$ and $u_2(x) = 1 - x$, $x \in [0, 1]$. After each session of bargaining, the utility functions of both players get decreased proportionally to δ.





Imagine that player *II* knows the alternative x chosen by the opponent at the next shot. The alternative is accepted, if he offers to player *I* an alternative y such that his utility $u_1(y)$ is not smaller than the utility at the next shot: $\delta u_1(x)$. This brings to the inequality $y \geq \delta x$. Furthermore, the maximal utility of player *II* is achieved under $y = \delta x$. Therefore, his optimal response to the opponent's strategy x makes up

$$x_2 = \delta x.$$

Sequential bargaining. Best response of player *I*

Now, suppose that player *I* knows the strategy x_2 selected by player *II* at the next shot. Then his offer y is accepted by player *II* at the current shot, if the corresponding utility $u_2(y)$ of player *II* appears not smaller than at the next shot (the quantity $\delta u_2(x_2)$). This condition is equivalent to the inequality $1 - y \geq \delta(1 - \delta x)$, or

$$y \leq 1 - \delta(1 - \delta x).$$

Hence, the best response of player *I* at the current shot makes up $x_1 = 1 - \delta(1 - \delta x)$. The solution x gives a subgame-perfect equilibrium in negotiations if $x_1 = x$, or $x = 1 - \delta(1 - \delta x)$. It follows that the Equilibrium is

$$x^* = \frac{1}{1 + \delta}.$$

There are n players who are negotiating the meeting time. The objective is to find a meeting time that satisfies all participants. The players' utilities are represented by linear unimodal functions $u_i(x), x \in [0, 1], i = 1, 2, ..., n$. The maximum values of the utility functions are located at the points $i/(n-1), i = 0, ..., n-1$.

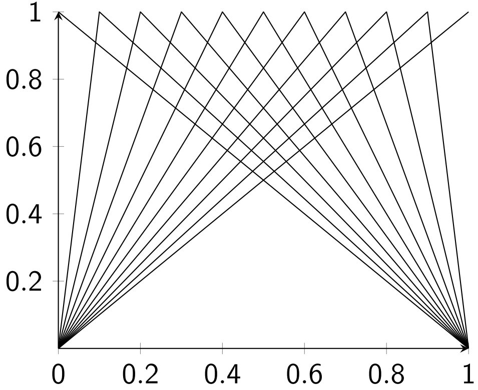


Fig. 4. Utility functions

Choosing the meeting time for n persons

Players take turns

$$1 \to 2 \to 3 \to \ldots \to (n-1) \to n \to 1 \to \ldots.$$

Discounting factor $\delta < 1$. After each negotiation session, the utility functions of all players will decrease proportionally to δ.
We will look for a solution in the class of **stationary strategies**, when it is assumed that the decisions of the players will not change during the negotiation time. This will allow us to limit ourselves to

$$1 \to 2 \to 3 \to \ldots \to (n-1) \to n \to 1.$$

We will use the **method of backward induction**.
To do this, assume that player n is looking for his best response, knowing player 1's proposal, then player $(n-1)$ is looking for his best responce, knowing player n's solution,, etc.
In the end, we find the best response of the player 1, and it should coincide with his offer at the beginning of the procedure.

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \ldots \leftarrow (n-1) \leftarrow n \leftarrow 1.$$

$$u_1(x) = x, u_2(x) = \begin{cases} 2x & x \in [0; \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}; 1], \end{cases}, u_3(x) = 1 - x.$$

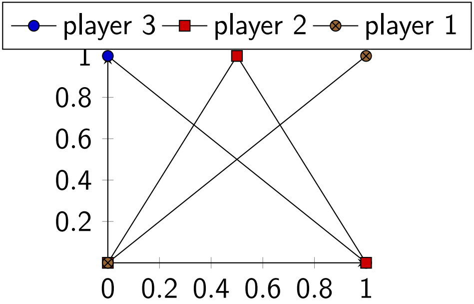


Fig.5. Utility functions

Solution for three persons. First player moves first

We use the backward induction

$$1 \leftarrow 2 \leftarrow 3 \leftarrow 1.$$

The first move from the end is **player 1**. Suppose, his choice is $x \in [\frac{1}{2}; 1]$.
Player 3 moves in front of player 1. Now he knows the proposition of player 1. His best response $y \in [0, 1]$ satisfies the following logic.

- For player 1 to accept the proposal y it is needed $u_1(y) \geqslant \delta u_1(x)$. That is equivalent to $y \geqslant \delta x$.
- For player 2 to accept the proposal y it is needed $u_2(y) \geqslant \delta u_2(x)$, That is equivalent to
$$\begin{cases} 2y \geqslant 2\delta(1-x), \\ 2(1-y) \geqslant 2\delta(1-x). \end{cases}$$

It gives the suitable interval for a proposal of player 3 $[\delta x; 1-\delta(1-x)]$, which satisfies both players 1 and 2. So, optimal proposition for player 3 is $\hat{x}_3 = \delta x$.

Player 2 moves in front of player 2. Now he knows the proposition of player 1 (x) and player 3 (δx). His best response $y \in [0,1]$ satisfies the following arguments.

- For player 1 to accept the proposal y it is needed $y \geqslant \delta^2 x$.
- For player 3 to accept the proposal y it is needed $1 - y \geqslant \delta(1 - \delta x)$.

Here suitable interval for player 2 is $[\delta^2 x; 1 - \delta(1 - \delta x)]$. Maximum utility for player 2 is the value $\frac{1}{2}$. Assume that

$$\frac{1}{2} \in [\delta^2 x; 1 - \delta(1 - \delta x)] \quad (1)$$

then the best response of player 2 is $\hat{x}_2 = \frac{1}{2}$.

Solution for three persons. First player moves first

Finally, **player 1** moves.

- Player 2 accepts his proposal x if $\begin{cases} 2x \geqslant \delta, \\ 2(1-x) \geqslant \delta. \end{cases}$
- Player 3 accepts his proposal x if $1 - x \geqslant \delta(1 - \frac{1}{2})$.

So, the suitable interval for the proposition of player 1 is $[\frac{\delta}{2}; 1 - \frac{\delta}{2}]$.
Utility function of player 1 is increasing. It yields the best proposition for player 1 $x^* = 1 - \frac{\delta}{2}$. Notice that for any δ it takes place $x^* \geqslant \frac{1}{2}$ (assumption (1)).
So, $x^* = 1 - \frac{\delta}{2}$ is the equilibrium for all $\delta \in [0; 1]$.

Solution for three persons. Second player moves first

Now suppose that the first move from the end is **player 2**.
The backward induction is determined by sequence

$$2 \leftarrow 3 \leftarrow 1 \leftarrow 2.$$

Assume that proposition of player 2 is $x = \frac{1}{2}$.
Player 1 knows the proposition of player 2. He makes an offer $y \in [0, 1]$, with the aim of satisfying the players 2 and 3 :

(Player 2) $\begin{cases} 2y \geqslant \delta, \\ 2(1-y) \geqslant \delta, \end{cases}$ (player 3) $1 - y \geqslant \delta(1 - \frac{1}{2})$.

The suitable interval $[\frac{\delta}{2}; 1 - \frac{\delta}{2}]$. It yields the best response of player 1 is

$$\hat{x}_1 = 1 - \frac{\delta}{2}.$$

That is the same decision when the player 1 moves first.

Solution for three persons. Second player moves first

Now **player 3** moves. He proposes $y \in [0,1]$. To be accepted by 1 and 2 it must be:

(Player 1) $y \geqslant \delta(1-\frac{\delta}{2})$, (player 2) $\begin{cases} 2y \geqslant 2\delta(1-1+\frac{\delta}{2}), \\ 2(1-y) \geqslant 2\delta(1-1+\frac{\delta}{2}). \end{cases}$

It yields $y \in [\delta(1-\frac{\delta}{2}); 1-\frac{\delta^2}{2}]$, and the best response of 3 is $\hat{x}_3 = \delta(1-\frac{\delta}{2})$

Player 2 moves. He makes an offer $y \in [0,1]$. To be accepted by 1 and 3 it must be

(Player 1) $y \geqslant \delta^2(1-\frac{\delta}{2})$, (player 3) $1-y \geqslant \delta(1-\delta(1-\frac{\delta}{2}))$.

Notice that $\frac{1}{2} \in [\delta^2(1-\frac{\delta}{2}); \delta(1-\delta(1-\frac{\delta}{2}))]$.

So, proposal $x^* = \frac{1}{2}$ is the solution of the game if he moves first.

Bargaining of n persons. Small δ

Let $\delta \leqslant \frac{1}{2}$ and player 1 moves first.

Theorem 1. *If*

$$\delta\left(1-\frac{\delta}{n-1}\right) \leqslant \frac{2}{n-1}, \qquad (1)$$

then the equilibrium is $x^* = 1 - \frac{\delta}{n-1}$.

Hint. The sequence of best responses is:

$$1 \to 1-\frac{\delta}{n-1},$$

$$2 \to \frac{n-2}{n-1},$$

$$3 \to \frac{n-3}{n-1},$$

$$\ldots$$

$$(n-1) \to \frac{1}{n-1},$$

$$n \to \delta\left(1-\frac{\delta}{n-1}\right),$$

Bargaining of n persons. Small δ

Таблица: Interval for δ where solution is of the form $x^* = 1 - \frac{\delta}{n-1}$

n	5	10	20	30	50
$(0, \delta)$	(0, 0.5)	(0, 0.2279)	(0, 0.1059)	(0, 0.0691)	(0, 0.0409)

Bargaining of n persons. Large δ

We consider the sequence

$$1 \to 2 \to 3 \to \cdots \to (n-1) \to n \to 1.$$

Theorem 2. *Let $n = 2k$ even number. If*

$$\frac{\delta^{k-1}}{1+\delta^k} \geqslant \frac{k-1}{2k-1}, \tag{2}$$

then the optimal proposal of player 1 is $x^* = \frac{1}{1+\delta^k}$.
Theorem 3. *Let $n = 2k+1$ is odd number. If*

$$\delta^k \geqslant \frac{k-1}{k}, \tag{3}$$

then the optimal proposal of player 1 is $x^* = 1 - \frac{1}{2}\delta^k$.

Bargaining of n persons.

When δ changes from 1 to 0, the optimal offer of player 1 changes from $\frac{1}{2}$ to 1. That is, when the value of δ is close to 1, the players have a lot of time to negotiate, so the offer of player 1 should be fair to everyone. If the discounting factor is close to 0, the utilities of the players decreases rapidly and they must quickly make a decision that is beneficial to player 1.

Consider the general case of n players. The utilities are described by continuous quasiconcave unimodal functions $u_i(x), i = 1, 2..., n$. Denote by $c_1, c_2, ..., c_n$ the maximum points of the utility functions. Players sequentially offer different alternatives; accepting an alternative requires the consensus of all participants. The sequence of moves is $1 \to 2 \to ... \to n \to 1 \to 2 \to ....$ We involve the same idea as in the case of three players.
Assume that player 1 announces his strategy x. Knowing this strategy, player n can compute his best response. And his offer y will be accepted by player j, if $u_j(y)$ appears not less than $\delta u_j(x)$; denote this set by $I_j(x)$. Note that, for any j, the set $I_j(x)$ is nonempty, so long as $x \in I_j(x)$. Since $u_j(x)$ is quasiconcave, $I_j(x)$ represents a closed interval. Consequently, there exists a closed interval $\Cap_{j=1}^{n-1} I_j(x)$, we designate it by $[a_n, b_n](x)$.

Bargaining on meeting time. The general case

Maximize the function $u_n(y)$ on the interval $[a_n, b_n](x)$. Actually, this is the best response of player n, and, by virtue of the above assumptions, it takes the form

$$x_n(x) = \begin{cases} a_n, & \text{if } a_n > c_n, \\ b_n, & \text{if } b_n < c_n, \\ c_n, & \text{if } a_n \leq c_n \leq b_n. \end{cases}$$

Now, imagine that player $n-1$ is informed of the strategy x_n to-be-selected by player n at the next shot. Similarly, he will make such offer y to player j, and this offer is accepted if $u_j(y) \geq \delta u_j(x_n)$, $j \neq n-1$. For each j, the set of such offers forms a closed interval; moreover, the intersection of all such intervals is nonempty and turns out a closed interval $[a_{n-1}, b_{n-1}](x_n)$.

Bargaining on meeting time. The general case

Again, maximize the function $u_{n-1}(y)$ on this interval. Actually, this maximum gets attained at the point

$$x_{n-1}(x_n) = \begin{cases} a_{n-1}, & \text{if } a_{n-1} > c_{n-1}, \\ b_{n-1}, & \text{if } b_{n-1} < c_{n-1}, \\ c_{n-1}, & \text{if } a_{n-1} \leq c_{n-1} \leq b_{n-1}. \end{cases}$$

Here x_{n-1} indicates the best response of player $n-1$ to the strategy x_n chosen by player n.
Following this line of reasoning, we finally arrive at the best response of player 1, *viz.*, the function $x_1(x_2)$.
By virtue of the assumptions, all constructed functions $x_i(x), i = 1, ..., n$ appear continuous. And the superposition of the mappings

$$x_1(...(x_{n-1}(x_n)...)(x)$$

is a continuous self-mapping of the closed interval $[0, 1]$.

Brouwer's fixed-point theorem claims that there exists a fixed point x^* such that

$$x_1(...(x_{n-1}(x_n)...)(x^*) = x^*.$$

Consequently, we have established the following result.

Theorem 4. *Bargaining of meeting time with continuous quasiconcave unimodal utility functions admit a subgame-perfect equilibrium.*

We consider the cake cutting problem with unit cake and n players.
There is another independent participant (an arbitrator).
The players decide to agree or disagree with them.
The ultimate decision is either by majority or complete consent.
Assume that the arbitrator represents a random generator.
Negotiations run on a given time interval K.
At each shot, the arbitrator makes random offers. Players observe their offers and accept or reject them.
Calculate the number of players satisfied by their offer; if this number turns out not less than a given threshold p, the offer is accepted. Otherwise, the offered alternative is rejected and players proceed to the next shot for considering another alternative.

The size of the cake is discounted by some quantity δ, where $\delta < 1$. If negotiations result in no decision, each player receives a certain portion b, where $b << 1/n$.
Let the random generator be described by the Dirichlet distribution with the density function

$$f(x_1, ..., x_n) = \frac{1}{B(k)} \prod_{i=1}^{n} x_i^{k_i - 1},$$

where $x_i \geq 0$, $\sum_{i=1}^{n} x_i = 1$ and $k_i \geq 1$. The constant $B(k)$ in this formula,

$$B(k) = B(k_1, ..., k_n) = \frac{\prod_{i=1}^{n} \Gamma(k_i)}{\Gamma(k_1 + ... + k_n)},$$

depends on a set of parameters $(k_1, ..., k_n)$. They serve for adjusting the weights of appropriate negotiators.

Stochastic Design in the Cake Cutting Problem. Three players

Bargaining cover the horizon of K shots.
Suppose that k shots remain. Players receive offers that form a vector (x_1^k, x_2^k, x_3^k). At each shot, offers represent random variables distributed according to the Dirichlet law. In other words, the joint density function takes the form

$$f(x_1, x_2, x_3) = \frac{\Gamma(k_1 + k_2 + k_3)}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)} x_1^{k_1 - 1} x_2^{k_2 - 1} x_3^{k_3 - 1},$$

where $x_1 + x_2 + x_3 = 1$. For a given offer vector (x_1, x_2, x_3), each player has to choose between two alternatives: (a) accepting a current offer (b) rejecting a current offer.
An allocation (x_1, x_2, x_3) takes place if all players agree at some shot. Otherwise, players move to nest shot $k - 1$. And the discounting effect reduces the size of the cake to $\delta \leq 1$.
The described process continues until all players support an offer or shot $k = 0$ comes.

Stochastic Design in the Cake Cutting Problem. Three players

Denote by H_k the value of this game when k shots remain to the end of negotiations. Suppose that each player is informed of his personal offer only. Let (x_1, x_2, x_3) specify the offers for players I, II, III, respectively. Since $x_1 + x_2 + x_3 = 1$, it suffices to handle the variables x_1, x_2.
First, study the symmetrical case of the Dirichlet distribution, where $k_1 = k_2 = k_3 = 1$:

$$f(x_1, x_2) = 2, \quad x_1 + x_2 \leq 1, \quad x_1, x_2 \geq 0.$$

Introduce the strategies $\mu_i(x_i)$, where $i = 1, 2, 3$. These are the probabilities that player i accepts a current offer x_i.

Stochastic Design in the Cake Cutting Problem. Three players

Theorem 5. *The optimal strategies of players at shot k have the form*

$$\mu_i(x_i) = I_{\{x_i \geq \delta H_{k-1}\}}, \qquad i = 1, 2, 3,$$

The value of this game satisfies the recurrent formulas

$$H_k = \delta H_{k-1} + \frac{1}{3}(1 - 3\delta H_{k-1})^3, \qquad H_0 = b.$$

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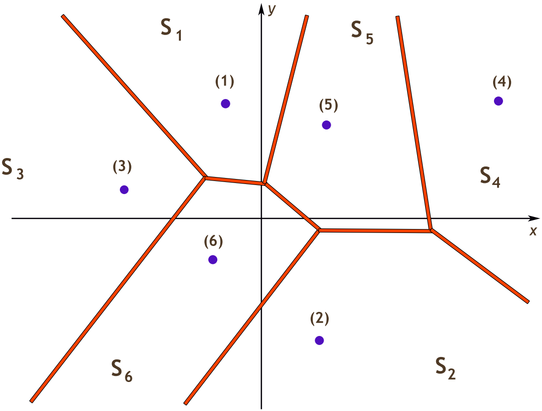
Models of Tournaments

Suppose that players $i \in N = \{1, 2, ..., n\}$ submit their projects for a tournament. Projects are characterized by a set of parameters $x^i = (x^i_1, ..., x^i_m)$. An arbitrator or arbitration committee considers the incoming projects and chooses a certain project by a stochastic procedure with a known probability distribution. The winner (player k) obtains the payoff $h_k(x^k)$, which depends on the parameters of his project.
Assume that project selection employs a multidimensional arbitration procedure choosing the project closest to the arbitrator's opinion.

A game-theoretic model of tournament organization

The decision of the arbitrator appears random. For a given set of projects $\{x^1, ..., x^n\}$, the set $S \subset R^m$ gets partitioned into n subsets $S_1, ..., S_n$ such that, if $a \in S_k$, then the arbitrator selects project k. The described partition is known as the Voronoi diagram.



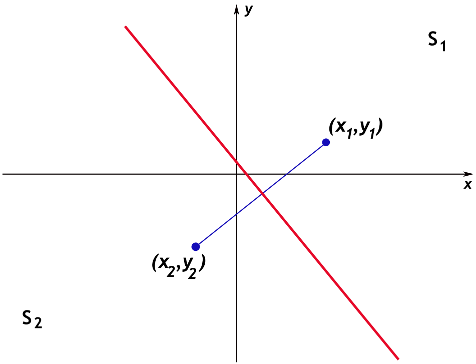
Therefore, the payoff of player k in this game can be defined through the mean value of his payoff as the arbitrator's decision hits the set S_k, i.e.,

$$H_k(x^1, ..., x^n) = \int_{S_k} h_k(x^k)\mu(dx_1, ..., dx_n) = h_k(x_k)\mu(S_k), k = 1, ..., n.$$

And so, we seek for a Nash equilibrium in this game—a strategy profile $x^* = (x^1, ..., x^n)$ such that

$$H_k(x^*||y^k) \leq H_k(x^*), \quad \forall y^k, \quad k = 1, ..., n.$$

Tournament for two projects with the Gaussian distribution



Player *I* strives for maximizing the sum $x + y$, whereas the opponent (player *II*) seeks to minimize it.
Arbitrator applies a procedure with the Gaussian distribution $f(x, y) = \frac{1}{2\pi} \exp\{-(x^2 + y^2)/2\}$.
Players submit their offers (x_1, y_1) and (x_2, y_2).

The space of arbitrator's decisions is divided into two subsets, S_1 and S_2.
Their boundary represents a straight line

$$y = -\frac{x_1 - x_2}{y_1 - y_2}x + \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(y_1 - y_2)}.$$

Therefore, the payoff of player I in this game acquires the form

$$H(x_1, y_1; x_2, y_2) = (x_1 + y_1)\mu(S_1) =$$

$$= (x_1+y_1)\int_R\int_R f(x,y)I\{y \geq -\frac{x_1 - x_2}{y_1 - y_2}x+\frac{(x_1^2 - x_2^2 + y_1^2 - y_2^2)}{2(y_1 - y_2)}\}dxdy.$$

Symmetry yields $x^2 = y^2 = -a$. Then

$$H(x_1, y_1) = (x_1+y_1)\int_R\int_R f(x,y)I\{y \geq -\frac{x_1 + a}{y_1 + a}x+\frac{(x_1^2 + y_1^2 - 2a^2)}{2(y_1 + a)}\}dxd$$

A game-theoretic model of tournament organization

The best response of player I satisfies the condition

$$\frac{\partial H}{\partial x_1} = 0, \frac{\partial H}{\partial y_1} = 0.$$

Note that, owing to symmetry, $\mu(S_1) = 1/2$ and $x_1 = y_1 = a$.

$$\frac{1}{2} - 2a \int_R \frac{1}{2\pi} \exp\{-\frac{1}{2}(x^2 + x^2)\} \frac{-x + a}{2a} dx = 0,$$

whence it follows that

$$\int_{-\infty}^{\infty} (-x + a) \frac{1}{2\pi} e^{-x^2} dx = \frac{1}{2}.$$

Finally, we obtain the optimal value

$$a = \sqrt{\pi}.$$

Therefore, the optimal strategies of players in this game consist in the offers $(-\sqrt{\pi}, -\sqrt{\pi})$ and $(\sqrt{\pi}, \sqrt{\pi})$, respectively.

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Thank you!